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THE PROBLEM OF AN ELASTIC CIRCULAR CYLINDER

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Abstract—The subject of this analysis is a homogeneous, isotropic and elastic circular cylinder subjected to prescribed forces or displacements at its surfaces. A general method of solution is presented with the help of which one can satisfy exactly arbitrary boundary conditions prescribed on the curved and flat surfaces of a hollow or solid cylinder of any length. Boundary conditions in displacements lead to the simplest presentation of the solution and are, therefore, used to demonstrate the method. It is considered sufficient to solve two fundamental problems and then to use linear superposition to solve any other specific problem. The first problem deals with satisfying arbitrary displacements on the curved surfaces of an infinitely long cylinder; while in the second problem arbitrary displacements on the flat end of a semi-infinite cylinder are satisfied with zero displacements maintained on its curved surfaces.

NOTATION

a, b	outer and inner radii of the cylinder
l	length of the cylinder
u, v, w	components of displacement of a point in the cylinder in the x, φ, and r directions respectively
E, ν	elastic constants
$f_1(r), g_1(r), h_1(r)$	functions determining the radial variation of u, v, w in the First Fundamental Problem
$\mathbf{u}$	vector containing u, v, w (or u and w in the axisymmetric deformation problem)
$\boldsymbol{\zeta}(r)$	vector containing f_1, g_1, h_1
$\mathbf{X}(x)$, $\mathbf{\Phi}(\varphi)$	3×3 diagonal matrices
λ, m	parameters
$(\ )'$	$= \mathrm{d}(\ )/\mathrm{d}r$
ρ	$= \lambda r/a$
γ	$= m+4(1-\nu)$
$\mathbf{\Delta}_1\left(\frac{\lambda r}{a}\right)$	matrix whose columns form sets of linearly independent solutions for f_1, g_1, and h_1
$\overline{\mathbf{\Delta}}(r)$	matrix containing solutions for f_1, g_1 and h_1 when $\lambda = 0$
$f_2(r), g_2(r), h_2(r)$	functions determining the radial variation of u, v, w in the Second Fundamental Problem
$\boldsymbol{\xi}(r)$	vector containing f_2, g_2, h_2 (or f_2 and h_2)
$\mathbf{\Delta}_2(\lambda r/a)$	matrix whose columns form sets of linearly independent solutions for f_2, g_2, h_2 (or f_2 and h_2)
$\mathbf{d}$	vector containing six (or four) constants d_i
$\mathbf{d}_n^*$	normalized vector
$\boldsymbol{\xi}_n(r)$	eigenvector corresponding to the eigenvalue λ_n
$\boldsymbol{\xi}_0(r)$	vector containing functions f_0, g_0, h_0 (or f_0, h_0) prescribed at $x = 0$
$\boldsymbol{\eta}(r)$	auxiliary vector containing three (or two) auxiliary functions
$\mathbf{y}(r)$	extended vector containing $\boldsymbol{\xi}(r)$ and $\boldsymbol{\eta}(r)$
$\boldsymbol{\eta}_0(r)$	arbitrarily prescribed vector
$\mathbf{y}_0(r)$	vector containing $\boldsymbol{\xi}_0(r)$ and $\boldsymbol{\eta}_0(r)$

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μ_1	$= 2(1-\nu)/(1-2\nu)$
$\mathbf{z}(r)$	adjoint of $\mathbf{y}(r)$
$\mathbf{T}(r)$	transformation matrix
$\mathbf{T}_0$	$= r\mathbf{T}$
$\bar{\mathbf{S}}$	$= \mathbf{T}^T\mathbf{B} = \mathbf{B}^T\mathbf{T}$
G_n	normalizing constant

1. INTRODUCTION

DETERMINING the state of stress and strain within a homogeneous, isotropic and elastic circular cylinder subjected to prescribed forces or displacements at its surfaces, is one of the classical problems of the mathematical theory of elasticity. The problem essentially reduces to finding solutions to the equations of elasticity in cylindrical coordinates and then adapting them to the prescribed boundary conditions at the curved and flat surfaces of the cylinder. The method of series, wherein the solutions are assumed in the form of a series of special functions, can be employed successfully to solve these equations. Pochhammer [1] and Chree [2] obtained very general solutions in this manner and expressed them in Fourier–Bessel series. Chree has illustrated the use of these solutions in satisfying various types of boundary conditions prescribed on the curved and flat surfaces of a solid cylinder. Filon [3] presented a detailed investigation of a number of problems of the equilibrium of a symmetrically loaded circular cylinder. However, neither of these authors was able to achieve a complete freedom of prescribing arbitrary stresses or displacements on all surfaces of the cylinder even considering only axisymmetric deformations.

Due to the practical importance of its solution, many authors [4–7] have since investigated this problem in various ways.† However, the solutions obtained are either approximate or they are suitable for satisfying a certain class of boundary conditions only. It is relatively simple to satisfy precisely the boundary conditions on the curved surfaces of an infinitely long cylinder. For a cylinder of finite length then, one could satisfy arbitrary boundary conditions on its curved surfaces precisely and in the light of Saint Venant's principle, boundary conditions on its flat ends could be satisfied in an approximate manner. Of course, the resulting solutions are not useful if one wants to study stresses or deformations near the ends of the cylinder.

In the present paper we propose to develop a completely general method of solution, with the help of which we are able to satisfy exactly any arbitrary boundary conditions prescribed on the curved and flat surfaces of a hollow or solid cylinder of any length. In order to be able to do so, we consider it sufficient to solve two fundamental problems separately. In the first, we shall consider an infinitely long cylinder (Fig. 1) which is subjected to known arbitrary boundary conditions on its inner and outer curved surfaces. We call this the First Fundamental Problem. The solution of this problem alone does not permit us the freedom of prescribing arbitrary boundary conditions on the flat ends of the cylinder. Hence, we must solve a second problem, in which we consider a semi-infinite cylinder subjected to homogeneous boundary conditions on its curved surfaces and to known arbitrary boundary conditions on its flat end. We call this the Second Fundamental Problem. If we superpose the solutions of the second problem on those of the first, this does not affect the boundary conditions prescribed on the curved surfaces in the first problem. This enables us

† Interested readers are also referred to the bibliography given by Lur'e [4] which lists relevant works of both Western and Russian writers.

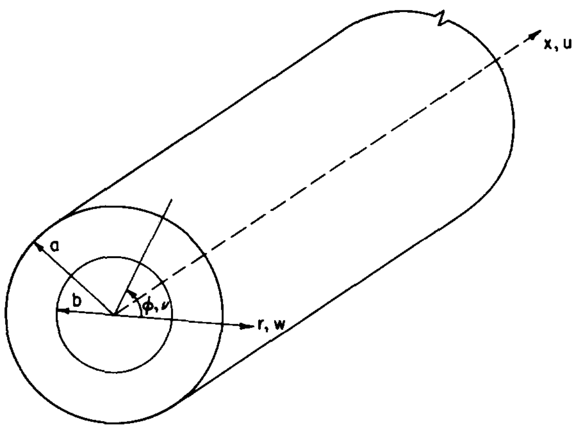

FIG. 1. Cylinder geometry.

to solve the general case of a cylinder of any length subjected to arbitrary boundary conditions on its curved and flat surfaces by suitably superposing the solutions of the two fundamental problems.

At each point of the curved and plane surfaces of the cylinder three boundary conditions must be prescribed. They may specify the components of the displacement or of the surface traction or a mixture of both. Since we shall develop differential equations for the displacements, boundary conditions in these quantities lead to the simplest presentation of the solution and will, therefore, be used here to demonstrate the method. It should, however, be noted, that in the case of prescribed surface tractions the procedure is substantially the same. The working out of details may be left for a later publication.

We shall derive partial differential equations governing the displacements u, v, w of a point in the cylinder and obtain their solutions suitable for the two problems. In the first problem, the constants of integration can then be determined easily from the displacements prescribed at the curved surfaces of the cylinder. The solution of the second problem is considerably more complicated. Here, the homogeneous boundary conditions on the curved surfaces $r = b$ and $r = a$ of the cylinder give rise to an infinite number of "eigensolutions". However, the differential equations governing the eigensolutions are not self-adjoint and also contain the parameter λ in a linear as well as nonlinear manner. Hence, the eigensolutions cannot be proved to form an orthogonal and complete set on (b, a). Thus, one is unable to expand directly arbitrarily prescribed displacements on the flat end of the cylinder, in a series of these eigensolutions. The only way to obtain such expansions would then seem to be some numerical technique. Lur'e [4], in solving the axisymmetric stress problem of a semi-infinite solid cylinder with zero loading on its curved surface, has tried to evaluate the unknown constants in such expansions by approximation in the "mean". However, this procedure involves many computational difficulties and hence becomes very limited in its use.

In this paper we shall employ a method† by which the constants in the series expansion of the end conditions can be determined in a simple manner. By introducing certain auxiliary functions (which are related to the original eigensolutions) we shall convert the original non-self-adjoint equations into an equivalent first-order equation for an extended

† Oral communication by Professor G. E. Latta.

eigenvector. The new equation, which is now linear in λ, can be shown to be in a self-adjoint form as defined by Bliss [8, 9]. Hence, orthogonality and completeness of the set of these eigenvectors can be deduced. With the help of these eigenvectors it is then possible to expand arbitrary displacements prescribed at the flat end of the cylinder in a series of the eigensolutions of our physical problem. This procedure is worked out in detail for the case of a cylinder under axisymmetric deformation, and can be extended easily to the general case of nonaxisymmetric deformation as indicated. It could also be applied to the plane stress problem of a semi-infinite rectangular plate subjected to homogeneous boundary conditions on its longitudinal edges and arbitrary boundary conditions on its narrow edge.

2. DERIVATION OF DIFFERENTIAL EQUATIONS

The material of the cylinder is assumed to be homogeneous, isotropic and linearly elastic following the generalized Hooke's law. With the assumptions that there are no body forces and that the deformations are small, the following equations can be written [10]:

Equilibrium equations:

$$\begin{aligned}
&\frac{\partial \tau_{rx}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{x\varphi}}{\partial \varphi}+\frac{\partial \sigma_x}{\partial x}+\frac{1}{r}\tau_{rx}=0\\
&\frac{\partial \tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\varphi}}{\partial \varphi}+\frac{\partial \tau_{x\varphi}}{\partial x}+\frac{2}{r}\tau_{r\varphi}=0\\
&\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi}+\frac{\partial \tau_{rx}}{\partial x}+\frac{\sigma_r-\sigma_\varphi}{r}=0.
\end{aligned} \tag{1a–c}$$

Kinematic relations:

$$\begin{aligned}
&\varepsilon_x=\frac{\partial u}{\partial x},\qquad \varepsilon_\varphi=\frac{w}{r}+\frac{1}{r}\frac{\partial v}{\partial \varphi},\qquad \varepsilon_r=\frac{\partial w}{\partial r}\\
&\gamma_{rx}=\frac{\partial u}{\partial r}+\frac{\partial w}{\partial x}\\
&\gamma_{x\varphi}=\frac{\partial v}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial \varphi}\\
&\gamma_{r\varphi}=\frac{1}{r}\frac{\partial w}{\partial \varphi}+\frac{\partial v}{\partial r}-\frac{v}{r}.
\end{aligned} \tag{2a–f}$$

Hooke's law:

$$\begin{aligned}
&\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x+\nu(\varepsilon_\varphi+\varepsilon_r)]\\
&\sigma_\varphi=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_\varphi+\nu(\varepsilon_r+\varepsilon_x)]
\end{aligned} \tag{3a, b}$$

$$\begin{aligned}
\sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + \nu(\varepsilon_x + \varepsilon_\varphi)] \\
\tau_{x\varphi} &= \frac{E}{2(1+\nu)}\gamma_{x\varphi} \\
\tau_{r\varphi} &= \frac{E}{2(1+\nu)}\gamma_{r\varphi} \\
\tau_{rx} &= \frac{E}{2(1+\nu)}\gamma_{rx}.
\end{aligned} \tag{3c–f}$$

Substituting equations (2) in (3) and introducing the resulting expressions for the stresses into equations (1), we obtain the following three partial differential equations for the displacements u, v, and w.

$$\frac{1-2\nu}{2}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right)+(1-\nu)\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 v}{\partial x\partial\varphi}+\frac{\partial^2 w}{\partial x\partial r}+\frac{1}{r}\frac{\partial w}{\partial x}\right)=0$$

$$\frac{1}{2r}\frac{\partial^2 u}{\partial x\partial\varphi}+\frac{1-2\nu}{2}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial x^2}\right)+\frac{1-\nu}{r^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{3-4\nu}{2}\frac{1}{r^2}\frac{\partial w}{\partial\varphi}+\frac{1}{2r}\frac{\partial^2 w}{\partial r\partial\varphi}=0$$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x\partial r}-\frac{3-4\nu}{2}\frac{1}{r^2}\frac{\partial v}{\partial\varphi}+\frac{1}{2r}\frac{\partial^2 v}{\partial r\partial\varphi}+(1-\nu)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{r^2}w\right)+\frac{1-2\nu}{2}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial^2 w}{\partial x^2}\right)=0. \tag{4a–c}$$

The solutions of these equations will contain integration constants which should be determined from displacements or stresses which may be prescribed at the curved and flat boundaries of the cylinder.

3. THE FIRST FUNDAMENTAL PROBLEM

We ask for a solution which is periodic in x with the period $2l$. Such a solution† is given by

$$\mathbf{u} = \mathbf{X}(x)\mathbf{\Phi}(\varphi)\boldsymbol{\zeta}(r) \tag{5}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} \cos\frac{\lambda x}{a} & 0 & 0 \\ 0 & \sin\frac{\lambda x}{a} & 0 \\ 0 & 0 & \sin\frac{\lambda x}{a} \end{bmatrix}, \tag{6a}$$

† Throughout this paper boldface capital letters denote matrices, boldface lower case letters columns (vectors).

$$\mathbf{\Phi}(\varphi) = \begin{bmatrix} \cos m\varphi & 0 & 0 \\ 0 & \sin m\varphi & 0 \\ 0 & 0 & \cos m\varphi \end{bmatrix}, \tag{6b}$$

$$\boldsymbol{\zeta}(r) = \begin{bmatrix} f_1(r) \\ g_1(r) \\ h_1(r) \end{bmatrix} \tag{6c}$$

and $\lambda = n\pi a/l$. When this is substituted in equations (4), the following system of simultaneous ordinary differential equations for the functions f_1, g_1, h_1 is found:

$$(1-2\nu)\left(f_1''+\frac{1}{r}f_1'-\frac{m^2}{r^2}f_1\right)-2(1-\nu)\frac{\lambda^2}{a^2}f_1+\frac{\lambda}{a}\frac{m}{r}g_1+\frac{\lambda}{a}(h_1'+\frac{1}{r}h_1) = 0 \tag{7a}$$

$$\frac{\lambda}{a}\frac{m}{r}f_1+(1-2\nu)\left(g_1''+\frac{1}{r}g_1'-\frac{1}{r^2}g_1-\frac{\lambda^2}{a^2}g_1\right)-2(1-\nu)\frac{m^2}{r^2}g_1-\frac{m}{r}h_1'-(3-4\nu)\frac{m}{r^2}h_1 = 0 \tag{7b}$$

$$-\frac{\lambda}{a}f_1'+\frac{m}{r}g_1'-(3-4\nu)\frac{m}{r^2}g_1+2(1-\nu)\left(h_1''+\frac{1}{r}h_1'-\frac{1}{r^2}h_1\right)-(1-2\nu)\left(\frac{m^2}{r^2}+\frac{\lambda^2}{a^2}\right)h_1 = 0 \tag{7c}$$

where $(\ )' = \mathrm{d}(\ )/\mathrm{d}r$. The system (7) is of the sixth order. It may be easily verified by substitution that it has the following linearly independent solutions:

regular solutions:

$$\begin{array}{llll} f_1 = I_m & 0 & \gamma I_m + \rho I_{m+1} \\ g_1 = I_{m+1} & -\frac{m}{\rho}I_m - I_{m+1} & 0 \\ h_1 = I_{m+1} & \frac{m}{\rho}I_m & \rho I_m \end{array} \tag{8}$$

singular solutions:

$$\begin{array}{llll} f_1 = K_m & 0 & \gamma K_m - \rho K_{m+1} \\ g_1 = -K_{m+1} & \frac{m}{\rho}K_m - K_{m+1} & 0 \\ h_1 = -K_{m+1} & -\frac{m}{\rho}K_m & \rho K_m \end{array} \tag{9}$$

where $\gamma = m+4(1-\nu)$ and I_m and K_m are the modified Bessel functions of the argument $\rho = \lambda r/a$. The solutions (8) and (9) can be written as the six columns of a 3×6 matrix which we shall denote by $\mathbf{\Delta}_1(\lambda r/a)$. A linear combination $\boldsymbol{\zeta}(r)$ of these solutions is obtained by postmultiplying $\mathbf{\Delta}_1$ by a column matrix $\mathbf{c}$ containing six constants $c_1, \ldots, c_6$. When we introduce this in equation (5) the solution for the displacements u, v, w takes the following form

$$\mathbf{u} = \mathbf{X}(x)\mathbf{\Phi}(\varphi)\mathbf{\Delta}_1\left(\frac{\lambda r}{a}\right)\mathbf{c}. \tag{10}$$

The unknown constants $c_1, \ldots, c_6$ appearing in equation (10) are to be determined from the displacements prescribed at the curved surfaces of the cylinder. We consider a typical set of terms in the Fourier expansion of the prescribed displacements as follows:

$$\text{At } r = a, \qquad \mathbf{u} = \mathbf{X}(x)\mathbf{\Phi}(\varphi)\mathbf{u}_{mn} \tag{11a}$$

$$\text{At } r = b, \qquad \mathbf{u} = \mathbf{X}(x)\mathbf{\Phi}(\varphi)\bar{\mathbf{u}}_{mn} \tag{11b}$$

where

$$\mathbf{u}_{mn} = \begin{bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{u}}_{mn} = \begin{bmatrix} \bar{u}_{mn} \\ \bar{v}_{mn} \\ \bar{w}_{mn} \end{bmatrix}$$

are the known amplitudes of displacements at the boundaries. Introducing equations (10) into (11), we arrive at the following set of six linear inhomogeneous equations for the unknown constants $\mathbf{c}$:

$$\mathbf{\Delta}_1(\lambda)\mathbf{c} = \mathbf{u}_{mn} \tag{12a}$$

$$\mathbf{\Delta}_1\left(\frac{\lambda b}{a}\right)\mathbf{c} = \bar{\mathbf{u}}_{mn}. \tag{12b}$$

After these equations have been solved for the constants $c_1, \ldots, c_6$, the displacements are given by equations (10) and the stresses may be obtained from equations (2) and (3).

It should be noted that

$$\det \begin{bmatrix} \mathbf{\Delta}_1(\lambda) \\ \mathbf{\Delta}_1\left(\frac{\lambda b}{a}\right) \end{bmatrix} \neq 0,$$

which implies that under zero loading we must have zero displacements.

In the special case $\lambda = 0$, solutions may be obtained by performing the proper limiting process in equations (8) and (9) (and their counterparts corresponding to interchange of sines and cosines in $\mathbf{X}(x)$) or by solving equations (7) directly, which in this case become equidimensional. We then obtain two linearly independent solutions for f_1 and four each for g_1 and h_1. We write them as the columns of a matrix $\bar{\mathbf{\Delta}}(r)$:

$$\bar{\mathbf{\Delta}}(r) = \begin{bmatrix} r^m & 0 & 0 & r^{-m} & 0 & 0 \\ 0 & r^{m-1} & -\gamma r^{m+1} & 0 & r^{-(m+1)} & (-\gamma+2m)r^{-(m-1)} \\ 0 & -r^{m-1} & (-\gamma+2m+2)r^{m+1} & 0 & r^{-(m+1)} & (\gamma-2)r^{-(m-1)} \end{bmatrix} \tag{13}$$

The corresponding constants $c_1, \ldots, c_6$ can be determined as before from the equations:

$$\bar{\mathbf{\Delta}}(a)\mathbf{c} = \mathbf{u}_{m0} \tag{14a}$$

$$\bar{\mathbf{\Delta}}(b)\mathbf{c} = \bar{\mathbf{u}}_{m0}. \tag{14b}$$

Thus, having solutions (8), (9) and (13), we are in a position to satisfy any arbitrary displacements prescribed on the curved surfaces of the cylinder. We shall next discuss a few special cases of interest.

Special cases

In the case of a solid cylinder which is free of singularities, we require that the displacements and stresses be finite at $r = 0$. The singular solutions (9) must then be dropped altogether. Therefore, $\mathbf{\Delta}_1(\lambda r/a)$ is now a 3×3 square matrix containing only the regular solutions (8). Furthermore, the general solution (10) for the displacements now contains only three unknown constants in $\mathbf{c}$, which can be determined as before from the three boundary conditions (11a) at $r = a$.

The problem of self-equilibrating loads applied to a cylindrical cavity in a medium extending to infinity in the radial direction can also be solved. In this case, we expect displacements and stresses which die out for large values of r. Hence we must only retain the singular terms (9) in our solution and solve for the three constants from the three boundary conditions at $r = b$.

When $m = 0$ in equations (5, 6), u and w become independent of φ and the circumferential displacement v vanishes. We thus obtain, as a particular case, the solution of the axisymmetric deformation problem. This is a forth-order problem and, correspondingly, four linearly independent solutions for f_1 and h_1 are obtained by substituting $m = 0$ in equations (8) and (9). The general solution for the displacements u and w then contains four constants which are determined from boundary conditions at $r = a$ and $r = b$. When in equations (5, 6) we interchange $\sin m\varphi$ and $\cos m\varphi$ and then put $m = 0$, displacements u and w vanish and v becomes independent of φ. Two linearly independent solutions for g_1 are then obtained by putting $m = 0$ in the solutions for g_1 in equations (8) and (9). The general solution for the displacement v now contains two constants, which are determined from the two boundary conditions at $r = a$ and $r = b$. The problem thus solved represents the twisting of a circular cylinder by loads applied at its curved surfaces.

4. SECOND FUNDAMENTAL PROBLEM

We now consider a semi-infinite cylinder subjected to zero displacements on its curved surfaces $r = a$ and $r = b$ and to arbitrary displacements at its flat end $x = 0$. In equations (4), we separate the variables by taking the solution in the form

$$\mathbf{u} = \mathbf{\Phi}(\varphi)\boldsymbol{\xi}(r)\,\mathrm{e}^{-\lambda x/a} \quad \text{where} \quad \boldsymbol{\xi}(r) = \begin{bmatrix} f_2(r) \\ g_2(r) \\ h_2(r) \end{bmatrix} \tag{15a, b}$$

and $\mathbf{\Phi}(\varphi)$ is defined by equation (6b). Introduction of equations (15) into equations (4) yields the following three equations for f_2, g_2 and h_2:

$$(1-2\nu)\left(f_2''+\frac{1}{r}f_2'-\frac{m^2}{r^2}f_2\right)+2(1-\nu)\frac{\lambda^2}{a^2}f_2-\frac{\lambda}{a}\frac{m}{r}g_2-\frac{\lambda}{a}\left(h_2'+\frac{1}{r}h_2\right) = 0 \tag{16a}$$

$$\frac{\lambda}{a}\frac{m}{r}f_2+(1-2\nu)\left(g_2''+\frac{1}{r}g_2'-\frac{1}{r^2}g_2+\frac{\lambda^2}{a^2}g_2\right)-2(1-\nu)\frac{m^2}{r^2}g_2-\frac{m}{r}h_2'-(3-4\nu)\frac{m}{r^2}h_2 = 0 \tag{16b}$$

$$-\frac{\lambda}{a}f_2'+\frac{m}{r}g_2'-(3-4\nu)\frac{m}{r^2}g_2+2(1-\nu)\left(h_2''+\frac{1}{r}h_2'-\frac{1}{r^2}h_2\right)-(1-2\nu)\left(\frac{m^2}{r^2}-\frac{\lambda^2}{a^2}\right)h_2 = 0. \tag{16c}$$

On solving these equations, we obtain the following six sets of linearly independent solutions:

regular solutions:

$$\begin{array}{llll} f_2 = J_m & 0 & \gamma J_m - \rho J_{m+1} & \\ g_2 = J_{m+1} & -\dfrac{m}{\rho} J_m + J_{m+1} & 0 & \qquad (17) \\ h_2 = J_{m+1} & \dfrac{m}{\rho} J_m & \rho J_m & \end{array}$$

singular solutions:

$$\begin{array}{llll} f_2 = Y_m & 0 & \gamma Y_m - \rho Y_{m+1} & \\ g_2 = Y_{m+1} & -\dfrac{m}{\rho} Y_m + Y_{m+1} & 0 & \qquad (18) \\ h_2 = Y_{m+1} & \dfrac{m}{\rho} Y_m & \rho Y_m & \end{array}$$

where J_m and Y_m are the Bessel functions of the argument $\rho = \lambda r/a$. These solutions may again be written as the columns of a 3×6 matrix, which we denote by $\mathbf{\Delta}_2(\lambda r/a)$. Any linear combination $\xi(r)$ can be written as the product of $\mathbf{\Delta}_2$ and a column $\mathbf{d}$ containing six constants $d_1, \ldots, d_6$. Hence

$$\mathbf{u} = \mathbf{\Phi}(\varphi)\mathbf{\Delta}_2(\lambda r/a)\,\mathbf{d}\,\mathrm{e}^{-\lambda x/a}. \qquad (19)$$

While the parameter λ appearing in the solutions (8), (9) of the First Fundamental Problem is known in advance, the parameter λ in equation (19) is an eigenvalue, which must be determined from the homogeneous boundary conditions on the curved surfaces. We consider in detail the case that zero displacements are prescribed:

$$\mathbf{u}(a, \varphi, x) \equiv 0, \qquad \mathbf{u}(b, \varphi, x) \equiv 0. \qquad (20a, b)$$

Upon introducing equation (19) into (20), we obtain a system of six linear equations for the elements of $\mathbf{d}$:

$$\mathbf{\Delta}_2(\lambda)\mathbf{d} = 0, \qquad \mathbf{\Delta}_2(\lambda b/a)\mathbf{d} = 0. \qquad (21a, b)$$

For a nontrivial solution $\mathbf{d}$ the coefficient determinant of these equations must vanish:

$$\det \begin{bmatrix} \mathbf{\Delta}_2(\lambda) \\ \mathbf{\Delta}_2(\lambda b/a) \end{bmatrix} = 0. \qquad (22)$$

This is a transcendental equation for λ, which occurs in the arguments of the Bessel functions and elsewhere. It has an infinite number of roots $\lambda_n (n = 1, 2, \ldots)$ which, in general, are complex. Of these we need only those which have a positive real part, since the solutions for our end load problem of a semi-infinite cylinder are expected to die out with increasing x.

For each eigenvalue λ_n (with positive real part), we can go back to equations (21) and solve for the ratios $d_{2,n}/d_{1,n}, \ldots, d_{6,n}/d_{1,n}$. The matrix $\mathbf{d}_n$ can then be written as $\mathbf{d}_n = d_n \mathbf{d}_n^*$

where $\mathbf{d}_n^*$ is a normalized column matrix $\{1, d_{2,n}/d_{1,n}, \ldots, d_{6,n}/d_{1,n}\}$. The corresponding solution for the displacements may now be written using equations (19):

$$\mathbf{u}_n = d_n \mathbf{\Phi}(\varphi)\mathbf{\Delta}_2(\lambda_n r/a)\mathbf{d}_n^* \, e^{-\lambda_n x/a},$$

and summation over all values of λ_n represents the general solution of the problem:

$$\mathbf{u}(r, \varphi, x) = \mathbf{\Phi}(\varphi) \sum_{n=1}^{\infty} d_n \mathbf{\Delta}_2(\lambda_n r/a)\, \mathbf{d}_n^* \, e^{-\lambda_n x/a}. \tag{23}$$

For each eigenvalue λ_n, the product $\mathbf{\Delta}_2(\lambda_n r/a)\,\mathbf{d}_n^*$ is a column matrix which satisfies equations (16) and homogeneous boundary conditions at $r = a$ and $r = b$. We denote this "eigenvector" by $\boldsymbol{\xi}_n(r)$. Then

$$\mathbf{u}(r, \varphi, x) = \mathbf{\Phi}(\varphi) \sum_{n=1}^{\infty} d_n \boldsymbol{\xi}_n(r)\, e^{-\lambda_n x/a}. \tag{24}$$

It may be noted that when $\lambda = 0$, the solutions (17), (18) degenerate to those given in equation (13). The corresponding six equations replacing equations (21) have a coefficient determinant which vanishes only when $a = b$, a case which is physically meaningless. Hence, as one would expect, $\lambda = 0$ is not an eigenvalue of our problem and we may disregard such a possibility in our further discussion.

Solution (24) contains an infinite number of unknown scalar constants d_n which must be determined from the boundary conditions at the flat end $x = 0$ of the cylinder. We choose to prescribe the three components of the displacement:

$$\mathbf{u}(r, \varphi, 0) = \mathbf{\Phi}(\varphi)\boldsymbol{\xi}_0(r) \quad \text{where} \quad \boldsymbol{\xi}_0(r) = \begin{bmatrix} f_0(r) \\ g_0(r) \\ h_0(r) \end{bmatrix}. \tag{25a, b}$$

Introducing equation (24) into (25a), we get

$$\boldsymbol{\xi}_0(r) = \sum_{n=1}^{\infty} d_n \boldsymbol{\xi}_n(r). \tag{26}$$

In this equation the vector $\boldsymbol{\xi}_0(r)$ which represents the prescribed displacements, contains real-valued functions. However, each term of the series on the right-hand sides of equations (24) and (26) is a vector whose elements are, in general, complex valued functions.

In order to determine the constants d_n in equation (26) we must be able to expand an arbitrary known vector $\boldsymbol{\xi}_0(r)$ in a series of eigenvectors $\boldsymbol{\xi}_n(r)$ on the interval (b, a). This would be a simple task if the vectors $\boldsymbol{\xi}_n(r)$ formed an orthogonal and complete set on the interval (b, a). This, unfortunately, is not true. Equations (16) are not self-adjoint and the parameter λ occurs in these equations as a linear as well as a nonlinear factor. The eigenvectors $\boldsymbol{\xi}_n(r)$ satisfying such differential equations and homogeneous boundary conditions, cannot be proved to form an orthogonal and complete set on (b, a).

At this stage, the only way to evaluate the constants d_n in (26) would then seem to be some numerical technique. Lur'e [4], for solving a corresponding axisymmetric ($m = 0$) stress problem, has obtained an equation similar to equation (26). He has then tried to separate each term of the series on the right-hand side into real and imaginary parts in order to obtain two linearly independent sets of functions with two sets of constants presumably required for the simultaneous representation of two arbitrary stresses at the end of the cylinder. The constants in the series are then evaluated by approximation in the "mean". This procedure not only involves considerable computational labor, but it also has the basic weakness that it cannot be extended to the case $m > 0$, in which the vectors $\boldsymbol{\xi}$ have three components. It is really not necessary to separate the right-hand side of equation (26) into real and imaginary parts since, as we shall see later, only one set of constants d_n is sufficient to prescribe three arbitrary displacements u, v, w (or u and w in the axisymmetric problem) at the end of the cylinder. We shall now present a method [11] by which the constants d_n in equation (26) can be determined in an indirect manner for arbitrary $\xi_0(r)$.

To do this we introduce an auxiliary vector $\boldsymbol{\eta}_n(r)$ (containing three elements) corresponding to each eigenvector $\boldsymbol{\xi}_n(r)$ and consider an extended vector

$$\mathbf{y}_n(r) = \begin{bmatrix} \boldsymbol{\xi}_n(r) \\ \boldsymbol{\eta}_n(r) \end{bmatrix} \tag{27}$$

wherein $\boldsymbol{\eta}_n(r)$ is chosen in such a way that the vector $\mathbf{y}_n(r)$ satisfies a first-order differential equation in which the parameter λ appears linearly. It can be shown that the differential equation and the boundary condition for $\mathbf{y}_n(r)$ are in a standard self-adjoint form as defined by Bliss [7, 8]. For such a system, the vectors $\mathbf{y}_n(r)$ can be shown to form an orthogonal† and complete set on (b, a), so that an arbitrary given vector $\mathbf{y}_0(r)$ can be represented on (b, a) by a series of $\mathbf{y}_n(r)$. We take

$$\mathbf{y}_0(r) = \begin{bmatrix} \boldsymbol{\xi}_0(r) \\ \boldsymbol{\eta}_0(r) \end{bmatrix} \tag{28}$$

where $\boldsymbol{\xi}_0(r)$ is the vector which we want to expand as in (26) while $\boldsymbol{\eta}_0(r)$ can be chosen freely. Thus for a given $\boldsymbol{\xi}_0(r)$ we can form different $\mathbf{y}_0(r)$ by choosing different $\boldsymbol{\eta}_0(r)$. However, for our problem, one convenient choice will suffice, for example $\boldsymbol{\eta}_0(r) \equiv \mathbf{0}$. After $\mathbf{y}_0(r)$ has been formed, the constants d_n in the series

$$\mathbf{y}_0(r) = \sum_{n=1}^{\infty} d_n \mathbf{y}_n(r) \tag{29}$$

can easily be obtained using the orthogonality of the $\mathbf{y}_n(r)$. From (27) and (28) we can see that equation (29) could also be written as

$$\boldsymbol{\xi}_0(r) = \sum_{n=1}^{\infty} d_n \boldsymbol{\xi}_n(r), \qquad \boldsymbol{\eta}_0(r) = \sum_{n=1}^{\infty} d_n \boldsymbol{\eta}_n(r).$$

The first equation is the same as (26). Thus by introducing an auxiliary vector $\boldsymbol{\eta}_n(r)$ we have succeeded in determining the constants d_n in (26) for a given $\boldsymbol{\xi}_0(r)$.

It should be emphasized here that for a chosen $\mathbf{y}_0(r)$ the constants d_n in (29) are uniquely determined because $\mathbf{y}_n(r)$ are an orthogonal and complete set of vectors on (b, a). However, for a given $\boldsymbol{\xi}_0(r)$ different choices of $\boldsymbol{\eta}_0(r)$ will lead to different sets of constants d_n from (29). This means that a given vector $\boldsymbol{\xi}_0(r)$ can be represented on (b, a) by a series of vectors $\boldsymbol{\xi}_n(r)$ in more than one way. However, for solving our problem, we need only one such expansion,

† Here, the orthogonality is meant in a generalized sense.

i.e., only one set of constants d_n in (26). After this has been obtained, the complete solution for the displacements is given by equations (24).

In the following section, the method discussed here is worked out in full detail for the complete solution of the Second Fundamental Problem for a cylinder under axisymmetric deformation.

5. SECOND FUNDAMENTAL PROBLEM—AXISYMMETRIC DEFORMATION

We now consider a semi-infinite circular cylinder subjected to zero displacements on its curved surfaces and arbitrary displacements at its flat end $x = 0$. Under axisymmetric deformation, the circumferential displacement v is zero and u, w are independent of φ. Equation (4b) is then identically satisfied while equations (4a, c) take the form:

$$\frac{1-2\nu}{2}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+(1-\nu)\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x \partial r}+\frac{1}{r}\frac{\partial w}{\partial x}\right) = 0 \tag{30a}$$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x \partial r}+(1-\nu)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{r^2}w\right)+\frac{1-2\nu}{2}\frac{\partial^2 w}{\partial x^2} = 0. \tag{30b}$$

Taking

$$\mathbf{u} = \boldsymbol{\xi}(r)\,\mathrm{e}^{-\lambda x/a} \tag{31a}$$

where now

$$\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}(r) = \begin{bmatrix} f_2(r) \\ h_2(r) \end{bmatrix} \tag{31b, c}$$

we obtain two equations for f_2, h_2 which we write in the following form:

$$(rf_2')' = -\frac{\lambda^2}{a^2}\frac{2(1-\nu)}{1-2\nu}rf_2+\frac{\lambda}{a}\frac{1}{1-2\nu}(rh_2'+h_2) \tag{32a}$$

$$(rh_2')' = -\frac{\lambda^2}{a^2}\frac{1-2\nu}{2(1-\nu)}rh_2+\frac{h_2}{r}+\frac{\lambda}{a}\frac{1}{2(1-\nu)}rf_2'. \tag{32b}$$

Solutions of these equations can be written directly by letting $m = 0$ in solutions (17) and (18). They are given below as the four columns of a matrix $\boldsymbol{\Delta}_2(\lambda r/a)$:

$$\boldsymbol{\Delta}_2\left(\frac{\lambda r}{a}\right) = \begin{bmatrix} J_0 & 4(1-\nu)J_0-\frac{\lambda r}{a}J_1 & Y_0 & 4(1-\nu)Y_0-\frac{\lambda r}{a}Y_1 \\ J_1 & \frac{\lambda r}{a}J_0 & Y_1 & \frac{\lambda r}{a}Y_0 \end{bmatrix} \tag{33}$$

where $J_0 = J_0(\lambda r/a)$ etc.

Following the same steps as in the preceding section, we obtain equations similar to (21) and (22) for the constants $d_1, \ldots, d_4$ and the eigenvalues λ_n respectively. As before, writing $\mathbf{d}_n = d_n \,.\, \mathbf{d}_n^*$ and summing over all values of λ_n (with positive real part), we arrive at the following solution for the displacements:

$$\mathbf{u}(r, x) = \sum_{n=1}^{\infty} d_n \boldsymbol{\xi}_n(r)\,\mathrm{e}^{-\lambda_n x/a} \tag{34}$$

where $\boldsymbol{\xi}_n(r) = \boldsymbol{\Delta}_2(\lambda_n r/a)\mathbf{d}_n^*$ is an eigenvector which satisfies the non-self-adjoint equations (32) and homogeneous boundary conditions ar $r = a$ and $r = b$. Let the arbitrarily prescribed displacements at $x = 0$ be given by

$$\mathbf{u}(r, 0) = \boldsymbol{\xi}_0(r) \quad \text{where now} \quad \boldsymbol{\xi}_0(r) = \begin{bmatrix} f_0(r) \\ h_0(r) \end{bmatrix}. \tag{35a, b}$$

Introducing equation (34) into (35) we get

$$\boldsymbol{\xi}_0(r) = \sum_{n=1}^{\infty} d_n \boldsymbol{\xi}_n(r). \tag{36}$$

To determine the unknown scalar constants d_n in equation (36), we now follow through, in detail, the procedure described in the preceding section.

First let us rewrite the two equations (32) conveniently as a single matrix equation† for the vector $\boldsymbol{\xi}(r)$.

$$(r\boldsymbol{\xi}')' = \lambda \mathbf{L}_1 \boldsymbol{\xi}' + \lambda^2 \mathbf{L}_2 \boldsymbol{\xi} + \lambda \mathbf{L}_3 \boldsymbol{\xi} + \mathbf{L}_4 \boldsymbol{\xi} \tag{37a}$$

where

$$\mathbf{L}_1 = \begin{bmatrix} 0 & \dfrac{1}{1-2\nu}\dfrac{r}{a} \\ \dfrac{1}{2(1-\nu)}\dfrac{r}{a} & 0 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -\dfrac{2(1-\nu)}{1-2\nu}\dfrac{r}{a^2} & 0 \\ 0 & -\dfrac{1-2\nu}{2(1-\nu)}\dfrac{r}{a^2} \end{bmatrix}$$
$$\mathbf{L}_3 = \begin{bmatrix} 0 & \dfrac{1}{a(1-2\nu)} \\ 0 & 0 \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{r} \end{bmatrix} \tag{37b–e}$$

do not depend on λ. The boundary conditions for $\boldsymbol{\xi}(r)$ are

$$\boldsymbol{\xi}(a) = 0 \quad \text{and} \quad \boldsymbol{\xi}(b) = 0. \tag{38a, b}$$

Now introduce an auxiliary vector

$$\boldsymbol{\eta}(r) = \begin{bmatrix} p(r) \\ q(r) \end{bmatrix}$$

containing two functions $p(r)$ and $q(r)$, such that

$$r\boldsymbol{\xi}' = \mathbf{P}\boldsymbol{\xi} + \mathbf{Q}\boldsymbol{\eta}, \qquad r\boldsymbol{\eta}' = \mathbf{R}\boldsymbol{\xi} + \mathbf{S}\boldsymbol{\eta}, \tag{39a, b}$$

where $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are 2×2 matrices. We use (39a) to eliminate the second derivative in (37a) and then use both equations (39) to eliminate $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}$. The resulting first-order differential

† Notations $\mathbf{Q}'$, $\mathbf{Q}^{-1}$ and $\mathbf{Q}^T$ denote respectively the differentiation with respect to r, the inverse and the transpose of a given matrix $\mathbf{Q}$.

equation for ξ becomes identically satisfied by any ξ if we require that the coefficients of ξ' and ξ vanish for all λ. (Of course, ξ and η have then still to satisfy equation (39)). This procedure yields the following two matrix differential equations:

$$\mathbf{P}+\mathbf{QSQ}^{-1}+r\mathbf{Q}'\mathbf{Q}^{-1}-\lambda\mathbf{L}_1 = 0, \tag{40a}$$

$$\mathbf{P}'+\frac{1}{r}\mathbf{QR}-\frac{1}{r}\mathbf{QSQ}^{-1}\mathbf{P}-\mathbf{Q}'\mathbf{Q}^{-1}\mathbf{P}-\lambda^2\mathbf{L}_2-\lambda\mathbf{L}_3-\mathbf{L}_4 = 0. \tag{40b}$$

It can be seen that in order to choose nontrivial matrices **P**, **Q**, **R**, **S** such that equations (40) are identically satisfied for all values of λ these unknown matrices must depend on λ. Looking at the form of equations (40), we can see that it is sufficient to consider these matrices to be linear in λ. We take them in the following form:

$$\mathbf{P} = \mathbf{P}_0+\lambda\mathbf{P}_1, \qquad \mathbf{Q} = \lambda\mathbf{Q}_1, \qquad \mathbf{R} = \lambda\mathbf{R}_1, \qquad \mathbf{S} = \mathbf{S}_0+\lambda\mathbf{S}_1. \tag{41a–d}$$

The new 2×2 matrices on the right hand side are the unknowns to be determined. To do this, we introduce equations (41) into (40) and equate to zero the coefficients of all powers of λ. This gives us the following five matrix differential equations:

$$\mathbf{P}_0+\mathbf{Q}_1\mathbf{S}_0\mathbf{Q}_1^{-1}+r\mathbf{Q}_1'\mathbf{Q}_1^{-1} = 0 \tag{42a}$$

$$\mathbf{P}_1+\mathbf{Q}_1\mathbf{S}_1\mathbf{Q}_1^{-1}-\mathbf{L}_1 = 0 \tag{42b}$$

$$\frac{1}{r}\mathbf{Q}_1\mathbf{R}_1-\frac{1}{r}\mathbf{Q}_1\mathbf{S}_1\mathbf{Q}_1^{-1}\mathbf{P}_1-\mathbf{L}_2 = 0 \tag{42c}$$

$$\mathbf{P}_1'-\frac{1}{r}\mathbf{Q}_1\mathbf{S}_0\mathbf{Q}_1^{-1}\mathbf{P}_1-\frac{1}{r}\mathbf{Q}_1\mathbf{S}_1\mathbf{Q}_1^{-1}\mathbf{P}_0-\mathbf{Q}_1'\mathbf{Q}_1^{-1}\mathbf{P}_1-\mathbf{L}_3 = 0 \tag{42d}$$

$$\mathbf{P}_0'-\frac{1}{r}\mathbf{Q}_1\mathbf{S}_0\mathbf{Q}_1^{-1}\mathbf{P}_0-\mathbf{Q}_1'\mathbf{Q}_1^{-1}\mathbf{P}_0-\mathbf{L}_4 = 0. \tag{42e}$$

Since these equations contain six unknown matrices $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{Q}_1$, $\mathbf{R}_1$, $\mathbf{S}_0$ and $\mathbf{S}_1$, their solution is not unique. For our purposes, any solution will do. A simple one was obtained by assuming the matrix $\mathbf{Q}_1$ to be diagonal and independent of r and then solving the somewhat simplified equations. The results are as follows:

$$\mathbf{P}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{P}_1 = \begin{bmatrix} 0 & \dfrac{1}{1-2\nu}\dfrac{r}{a} \\ 0 & 0 \end{bmatrix} \tag{43a, b}$$

$$\mathbf{Q}_1 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \dfrac{1}{\mu_1} \end{bmatrix}, \qquad \mathbf{R}_1 = \begin{bmatrix} -\dfrac{r^2}{a^2} & 0 \\ 0 & \dfrac{2\nu}{1-2\nu}\mu_1\dfrac{r^2}{a^2} \end{bmatrix} \tag{43c, d}$$

$$\mathbf{S}_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{S}_1 = \begin{bmatrix} 0 & 0 \\ \dfrac{\mu_1}{1-2\nu}\dfrac{r}{a} & 0 \end{bmatrix} \tag{43e, f}$$

where $\mu_1 = 2(1-\nu)/(1-2\nu)$.

These matrices look so simple that further efforts to get a simpler solution of equations (42) are unwarranted.

Using equations (43), the matrices **P**, **Q**, **R**, **S** are completely determined for each λ from equations (41). Now equations (39) can be written as a single equation for an extended vector $\mathbf{y}(r)$, when we substitute equations (41) in them. We get

$$r\mathbf{y}' = \mathbf{A}\mathbf{y} + \lambda\mathbf{B}\mathbf{y} \tag{44a}$$

where

$$\mathbf{y}(r) = \begin{bmatrix} \boldsymbol{\xi}(r) \\ \boldsymbol{\eta}(r) \end{bmatrix} = \begin{bmatrix} f_2(r) \\ h_2(r) \\ p(r) \\ q(r) \end{bmatrix} \tag{44b}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{44c}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{Q}_1 \\ \mathbf{R}_1 & \mathbf{S}_1 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{1-2\nu}\dfrac{r}{a} & \mu_1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu_1} \\ -\dfrac{r^2}{a^2} & 0 & 0 & 0 \\ 0 & \dfrac{2\nu}{1-2\nu}\mu_1\dfrac{r^2}{a^2} & \dfrac{\mu_1}{1-2\nu}\dfrac{r}{a} & 0 \end{bmatrix} \tag{44d}$$

We have now succeeded in converting our basic equation (37a) into equation (44a) for the extended vector $\mathbf{y}(r)$. The matrix equation (44a) represents a system of coupled first-order equations, in which the parameter λ occurs linearly. Boundary value problems for such equations have been treated in detail by Bliss [7, 8]. Following his treatment, we shall deduce orthogonality relations for any two extended vectors $\mathbf{y}_n$ and $\mathbf{y}_m$ corresponding to two different values λ_n and λ_m of λ.

Before doing this we must note that the auxiliary vector $\boldsymbol{\eta}(r)$ depends on $\boldsymbol{\xi}(r)$ and λ and can be determined from equations (39a) and (41):

$$\boldsymbol{\eta}(r) = \frac{r}{\lambda}\mathbf{Q}_1^{-1}\boldsymbol{\xi}' - \frac{1}{\lambda}\mathbf{Q}_1^{-1}\mathbf{P}_0\boldsymbol{\xi} - \mathbf{Q}_1^{-1}\mathbf{P}_1\boldsymbol{\xi}. \tag{45a}$$

Substitution from equations (43) and (31c) enables us to write explicitly:

$$\boldsymbol{\eta}(r) = \begin{bmatrix} p(r) \\ \\ q(r) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\mu_1}\dfrac{r}{\lambda}f_2' - \dfrac{1}{\mu_1}\dfrac{1}{(1-2\nu)}\dfrac{r}{a}h_2 \\ \\ \mu_1\dfrac{r}{\lambda}h_2' - \mu_1\dfrac{1}{\lambda}h_2 \end{bmatrix} \tag{45b}$$

hence

$$\mathbf{y}(r) = \begin{bmatrix} \xi(r) \\ \eta(r) \end{bmatrix} \begin{bmatrix} f_2 \\ h_2 \\ \frac{1}{\mu_1}\frac{r}{\lambda}f_2' - \frac{1}{\mu_1}\frac{1}{(1-2\nu)}\frac{r}{a}h_2 \\ \mu_1\frac{r}{\lambda}h_2' - \mu_1\frac{1}{\lambda}h_2 \end{bmatrix}. \tag{46}$$

For every known eigenvector $\xi_n(r)$ satisfying equation (37a) and boundary condition (38) of our physical problem, we can form from equations (46) an extended vector $\mathbf{y}_n(r)$ which satisfies equation (44a). Boundary conditions satisfied by $\mathbf{y}_n(r)$ can be written in the following form, using equations (38):

$$\mathbf{M}_1\mathbf{y}(a) + \mathbf{N}_1\mathbf{y}(b) = 0 \tag{47a}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{47b, c}$$

Orthogonality relations for $\mathbf{y}(r)$

We are considering a system

$$r\mathbf{y}' = \mathbf{A}\mathbf{y} + \lambda\mathbf{B}\mathbf{y} \tag{44a}$$

repeated

$$\mathbf{M}_1\mathbf{y}(a) + \mathbf{N}_1\mathbf{y}(b) = 0 \tag{47a}$$

repeated

wherein $\mathbf{A}, \mathbf{B}, \mathbf{M}_1, \mathbf{N}_1$ are all real and continuous on $b \leq r \leq a$; while λ and $\mathbf{y}$ will in general be complex. A system adjoint to equations (44a) and (47a) has the form

$$(r\mathbf{z})' = -\mathbf{A}^T\mathbf{z} - \lambda\mathbf{B}^T\mathbf{z} \tag{48a}$$

and

$$\mathbf{M}_2^T\mathbf{z}(a) + \mathbf{N}_2^T\mathbf{z}(b) = 0 \tag{48b}$$

where $\mathbf{M}_2$ and $\mathbf{N}_2$ are two matrices which can be easily chosen so that

$$\mathbf{M}_1\mathbf{M}_2 - \mathbf{N}_1\mathbf{N}_2 = 0. \tag{48c}$$

The boundary value problem defined by equations (44a) and (47a) is called "self-adjoint" if the differential equations and the boundary conditions of its adjoint (48) are equivalent to its own under a non-singular transformation $\mathbf{z} = \mathbf{T}\mathbf{y}$ (see Bliss [8, 9]). When we substitute for $\mathbf{z}$ in equations (48) and compare with equations (44a) and (47a), we get the following equations for determining the transformation matrix $\mathbf{T}$:

$$r\mathbf{T}'+\mathbf{A}^T\mathbf{T}+\mathbf{T}\mathbf{A}+\mathbf{T} = 0, \qquad \mathbf{B}^T\mathbf{T}+\mathbf{T}\mathbf{B} = 0 \tag{49a, b}$$

$$\mathbf{M}_2^T\mathbf{T} = \mathbf{\Lambda}\mathbf{M}_1, \qquad \mathbf{N}_2^T\mathbf{T} = \mathbf{\Lambda}\mathbf{N}_1$$

where $\mathbf{\Lambda}$ is a nonsingular matrix containing arbitrary constants. The last two equations may be combined into a single equation by using equation (48c):

$$\mathbf{M}_1\mathbf{T}^{-1}\mathbf{M}_1^T-\mathbf{N}_1\mathbf{T}^{-1}\mathbf{N}_1^T = 0. \tag{49c}$$

When we substitute values of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}_1$, $\mathbf{N}_1$ in equations (49), it can be seen that all three equations can be satisfied by taking

$$\mathbf{T} = \frac{1}{r}\mathbf{T}_0 \tag{50a}$$

with

$$\mathbf{T}_0 = \begin{bmatrix} 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & -1 \\ -\mu_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{50b}$$

It should be noted that the matrix $\mathbf{T}(r)$ is real, single valued and has continuous derivatives on $b \leq r \leq a$, $(b \neq 0)$. It is also skew-symmetric and nonsingular.

Let us now consider two vectors, $\mathbf{y}_m$ and $\mathbf{z}_n$ corresponding to two distinct values λ_m and λ_n, respectively, of λ. Premultiplying equation (44a) by $\mathbf{z}_n^T$ and postmultiplying the transposed equation (48a) by $\mathbf{y}_m$ we get the following:

$$\mathbf{z}_n^T(r\mathbf{y}_m') = \mathbf{z}_n^T\mathbf{A}\mathbf{y}_m+\lambda_m\mathbf{z}_n^T\mathbf{B}\mathbf{y}_m$$

and

$$(r\mathbf{z}_n^T)'\mathbf{y}_m = -\mathbf{z}_n^T\mathbf{A}\mathbf{y}_m-\lambda_n\mathbf{z}_n^T\mathbf{B}\mathbf{y}_m.$$

Addition of these two equations gives

$$(r\mathbf{z}_n^T\mathbf{y}_m)' = (\lambda_m-\lambda_n)\mathbf{z}_n^T\mathbf{B}\mathbf{y}_m.$$

If we now integrate with respect to r from b to a and substitute $\mathbf{z}_n = \mathbf{T}\mathbf{y}_n = (1/r)\mathbf{T}_0\mathbf{y}_n$, we get

$$(\lambda_m-\lambda_n)\int_b^a \mathbf{y}_n^T\mathbf{T}^T\mathbf{B}\mathbf{y}_m\,\mathrm{d}r = [\mathbf{y}_n^T\mathbf{T}_0^T\mathbf{y}_m]_{r=b}^a. \tag{51}$$

On expanding the right hand side and substituting boundary conditions (47) or (38), one can see that it vanishes and hence for $m \neq n$,

$$\int_b^a \mathbf{y}_n^T\bar{\mathbf{S}}\mathbf{y}_m\,\mathrm{d}r = 0 \tag{52a}$$

where we have used the notation

$$\overline{\mathbf{S}} = \mathbf{T}^T\mathbf{B} = \mathbf{B}^T\mathbf{T} = \begin{bmatrix} \frac{\mu_1 r}{a^2} & 0 & 0 & 0 \\ 0 & \frac{2\nu}{1-2\nu}\mu_1\frac{r}{a^2} & \frac{\mu_1}{a(1-2\nu)} & 0 \\ 0 & \frac{\mu_1}{a(1-2\nu)} & \frac{\mu_1^2}{r} & 0 \\ 0 & 0 & 0 & -\frac{1}{\mu_1 r} \end{bmatrix} \tag{52b}$$

Thus we have shown that any two distinct vectors $\mathbf{y}_m$ and $\mathbf{y}_n$ are "orthogonal" on (b, a) with respect to a "weight function" matrix $\overline{\mathbf{S}}(r)$. It should be noted that $\overline{\mathbf{S}}$ is real, symmetric and nonsingular.

Expansion of an arbitrary vector

Having obtained a complete orthogonal set† of vectors $\mathbf{y}_n$, we are in a position to expand an arbitrary given vector $\mathbf{y}_0$ on (b, a) in a series of vectors $\mathbf{y}_n$. Let

$$\mathbf{y}_0 = \sum_{n=1}^{\infty} d_n \mathbf{y}_n(r). \tag{53a}$$

Then, using orthogonality relations (52), we have constants d_n in the series given by:

$$d_n = \frac{1}{G_n}\int_b^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_0 \, \mathrm{d}r \tag{53b}$$

where G_n are the "normalizing constants:"

$$G_n = \int_b^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_n \, \mathrm{d}r. \tag{53c}$$

Constants d_n and G_n will, in general, be complex.

Now we can go back to our physical problem in which we wanted to expand a given arbitrary vector $\xi_0(r)$ in a series (36) of eigenvectors $\xi_n(r)$. For every $\xi_n(r)$, a corresponding vector $\mathbf{y}_n(r)$ is known from equations (46). Also for a given $\xi_0(r)$ we can choose $\eta_0(r)$ arbitrarily and form an extended vector $\mathbf{y}_0(r)$, which can be expanded into the series (53a). Hence the constants d_n in the series (36) are completely determined from equations (53b, c). Here one should recollect what was emphasized earlier; namely, that the constants d_n in expansion (36) are not uniquely determined for a given $\xi_0(r)$. They depend on $\mathbf{y}_0$ which can be formed in more than one way by choosing $\eta_0(r)$ differently each time. For our purpose it is sufficient to have only one such set of constants which can be determined by choosing $\eta_0(r)$ most conveniently (zero, for instance). After this has been obtained, the complete solution for the displacements of the cylinder is given by equations (34).

We are thus able to solve the Second Fundamental Problem by expanding the displacements prescribed at the end of the cylinder into a series of eigenvectors. The constants

† Theorems on completeness of a set of such vectors, etc. are not the subject matter of this work and the reader wishing to look more deeply into these matters is referred to Bliss [8, 9] who has proved similar theorems for "definitely" self-adjoint boundary value problems.

in this series can be obtained from equations (53b) by simple integration. Further, we shall obtain in the following sections a general expression from which the normalizing constants G_n can be obtained directly without any integration.

Orthogonality in the case of a solid cylinder

The derivation of the orthogonality relation (52) needs some modification when we consider a solid cylinder. In this case, equation (51) becomes

$$(\lambda_m - \lambda_n) \int_0^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_m \, dr = [\mathbf{y}_n^T \mathbf{T}_0^T \mathbf{y}_m]_{r=0}^a.$$

At $r = a$ we still prescribe zero displacements, hence equation (38a) still holds true. However at $r = 0$ we only require that the displacements and their first derivatives be finite. Substituting from equations (46) and (50) in the right-hand side and expanding we see that all terms vanish except one. We get

$$(\lambda_m - \lambda_n) \int_0^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_m \, dr = -\frac{\lambda_m - \lambda_n}{\lambda_m \lambda_n} \mu_1 h_{2n}(0) h_{2m}(0). \tag{54}$$

Solutions for $h_2(\lambda r/a)$ are found in the second row of the matrix $\mathbf{\Delta}_2(\lambda r/a)$ in equation (33). For the solid cylinder we have to consider only the two regular solutions from these and it can be seen that they both vanish when $r = 0$. Substituting this in the above equation we can once again establish the orthogonality relation (52) in the case of a solid cylinder. Due to the presence of a factor $(\lambda_m - \lambda_n)$ on both sides of equation (54) one is led to suspect that the integral on the left-hand side of this equation may be zero also when $m = n$. However, this integral for $m = n$ (i.e., the normalizing constant) can be evaluated and checked to be non-zero. This has been done later, for the example of a solid cylinder, by using the general expression for the normalizing constants derived in the next section.

Normalizing constants

To evaluate the constants G_n in equation (53c) directly without integration, consider two solutions $\mathbf{y}(r, \lambda)$ and $\mathbf{y}(r, \mu)$ of equation (44a) corresponding to two distinct values λ and μ of the parameter λ, not necessarily eigenvalues. Let these solutions also satisfy the boundary conditions:

$$\mathbf{N}_1 \mathbf{y}(b, \lambda) = 0, \qquad \mathbf{N}_1 \mathbf{y}(b, \mu) = 0.$$

This implies that the corresponding vectors $\boldsymbol{\xi}(r, \lambda)$ and $\boldsymbol{\xi}(r, \mu)$ satisfy equation (37a) and only one homogeneous boundary condition (38b) viz:

$$\boldsymbol{\xi}(b, \lambda) = 0 \quad \text{and} \quad \boldsymbol{\xi}(b, \mu) = 0.$$

Hence, these two vectors do not correspond to the eigenvectors of our problem but will do so only when they also satisfy the second homogeneous boundary condition at $r = a$ (equation (38a)). The vector $\mathbf{y}(r, \mu)$ satisfies the equation:

$$r\mathbf{y}'(r, \mu) = \mathbf{A}\mathbf{y}(r, \mu) + \mu \mathbf{B}\mathbf{y}(r, \mu).$$

Also from equation (48a) we can write

$$[r\mathbf{z}(r, \lambda)]' = -\mathbf{A}^T \mathbf{z}(r, \lambda) - \lambda \mathbf{B}^T \mathbf{z}(r, \lambda)$$

as the adjoint of the equation satisfied by $\mathbf{y}(r, \lambda)$. Premultiplying the first equation by $\mathbf{z}^T(r, \lambda)$ and postmultiplying the transposed second equation by $\mathbf{y}(r, \mu)$ and adding, we arrive at the following equation which corresponds to equation (51).

$$(\mu-\lambda)\int_b^a \mathbf{y}^T(r, \lambda)\mathbf{\bar{S}}y(r, \mu)\,\mathrm{d}r = [\mathbf{y}^T(r, \lambda)\mathbf{T}_0^T\mathbf{y}(r, \mu)]_{r=b}^{r=a}.$$

Expanding the right-hand side by substituting for $\mathbf{T}_0$ and $\mathbf{y}$ from equations (50b) and (46) and using the boundary conditions at $r = b$ one can see that it vanishes at the lower limit.† Dividing the remaining terms by $(\mu-\lambda)$ for $\mu \neq \lambda$ we get:

$$\begin{aligned}\int_b^a \mathbf{y}^T(r, \lambda)\mathbf{\bar{S}}\mathbf{y}(r, \mu)\,\mathrm{d}r = \frac{1}{\mu-\lambda}\bigg[&\frac{a}{\lambda}\{f_2'(a, \lambda)f_2(a, \mu)-\mu_1 h_2'(a, \lambda)h_2(a, \mu)\}\\ &+\frac{a}{\mu}\{-f_2(a, \lambda)f_2'(a, \mu)+\mu_1 h_2(a, \lambda)h_2'(a, \mu)\}\\ &-\frac{1}{1-2\nu}\{h_2(a, \lambda)f_2(a, \mu)-f_2(a, \lambda)h_2(a, \mu)\}\bigg]+\frac{1}{\lambda\mu}\mu_1 h_2(a, \lambda)h_2(a, \mu).\end{aligned}$$

Now taking the limit as $\mu \to \lambda$ and using L'Hospital's rule to evaluate this limit on the right hand side, we get

$$\begin{aligned}\int_b^a \mathbf{y}^T(r, \lambda)\mathbf{\bar{S}}\mathbf{y}(r, \lambda)\,\mathrm{d}r = \frac{a}{\lambda}\bigg\{&f_2'(a, \lambda)\frac{\partial}{\partial\lambda}f_2(a, \lambda)-\mu_1 h_2'(a, \lambda)\frac{\partial}{\partial\lambda}h_2(a, \lambda)\\ &-f_2(a, \lambda)\frac{\partial}{\partial\lambda}f_2'(a, \lambda)+\mu_1 h_2(a, \lambda)\frac{\partial}{\partial\lambda}h_2'(a, \lambda)\bigg\}\\ &-\frac{1}{1-2\nu}\bigg\{h_2(a, \lambda)\frac{\lambda}{\partial\lambda}f_2(a, \lambda)-f_2(a, \lambda)\frac{\partial}{\partial\lambda}h_2(a, \lambda)\bigg\}\\ &+\frac{1}{\lambda}\mu_1 h_2(a, \lambda)\frac{\partial}{\partial\lambda}h_2(a, \lambda).\end{aligned}$$

This equation is valid for any value of λ not necessarily an eigenvalue. When λ is an eigenvalue λ_n of the problem then $\boldsymbol{\xi}(r, \lambda_n)$ must satisfy the homogeneous boundary condition at $r = a$, i.e., $\boldsymbol{\xi}(a, \lambda_n) = \mathbf{0}$. Substituting this in the preceding equation, and writing $\mathbf{y}_n(r)$ for $\mathbf{y}(r, \lambda_n)$ etc., we get

$$G_n = \int_b^a \mathbf{y}_n^T\mathbf{\bar{S}}\mathbf{y}_n\,\mathrm{d}r = \frac{a}{\lambda_n}f_{2n}'(a)\left[\frac{\partial}{\partial\lambda}f_2(a, \lambda)\right]_{\lambda=\lambda_n}-\frac{a}{\lambda_n}\mu_1 h_{2n}'(a)\left[\frac{\partial}{\partial\lambda}h_2(a, \lambda)\right]_{\lambda=\lambda_n}. \quad (55)$$

On the right hand side of this equation $f_2(a, \lambda)$ and $h_2(a, \lambda)$ are the values at $r = a$ of the two functions $f_2(r, \lambda)$ and $h_2(r, \lambda)$ forming the vector $\boldsymbol{\xi}(r, \lambda)$ which satisfies the differential equation (37a) and only one boundary condition $\boldsymbol{\xi}(b, \lambda) = 0$. Further, when $\lambda = \lambda_n$, then $\boldsymbol{\xi}(r, \lambda_n)$ satisfies the second boundary condition $\boldsymbol{\xi}(a, \lambda_n) = 0$, i.e., it coincides with the eigenvector $\boldsymbol{\xi}_n$.

† In the case of a solid cylinder the right hand side can be shown to vanish at the lower limit $r = 0$ by the same reasoning as was used in the last section.

With the help of equation (55) the evaluation of the normalizing constants becomes a simple matter. In the next section, we shall use this equation to get the constants G_n for a solid cylinder.

Example

We illustrate the method of solution discussed so far, by solving a problem of a semi-infinite solid cylinder under axisymmetric deformation. The cylinder is subjected to zero displacements at its curved surface $r = a$ and the following displacements are prescribed at its flat end $x = 0$:

$$u(r, 0) = f_0(r) = r(1 - r/a) \qquad w(r, 0) = h_0(r) = 0. \tag{56a, b}$$

The solution for displacements consists only of the regular solutions and can be written down explicitly as follows by taking the first two columns of the matrix $\Delta_2(\lambda r/a)$ in equation (33),

$$\begin{aligned} u(x, r) &= \left[d_1 J_0\left(\frac{\lambda r}{a}\right) + d_2\left\{4(1-v)J_0\left(\frac{\lambda r}{a}\right) - \frac{\lambda r}{a} J_1\left(\frac{\lambda r}{a}\right)\right\}\right] e^{-\lambda x/a} \\ w(x, r) &= \left[d_1 J_1\left(\frac{\lambda r}{a}\right) + d_2 \frac{\lambda r}{a} J_0\left(\frac{\lambda r}{a}\right)\right] e^{-\lambda x/a} \end{aligned} \tag{57a, b}$$

where d_1 and d_2 are unknown constants. The boundary conditions at $r = a$ give us two linear homogeneous equations for d_1 and d_2:

$$d_1 J_0(\lambda) + d_2\{4(1-v)J_0(\lambda) - \lambda J_1(\lambda)\} = 0, \qquad d_1 J_1(\lambda) + d_2 \lambda J_0(\lambda) = 0. \tag{58a, b}$$

For non-trivial solutions of d_1 and d_2, we get the following transcendental equation for the eigenvalues λ_n:

$$\lambda_n j_0^2 + \lambda_n j_1^2 - 4(1-v) j_0 j_1 = 0 \tag{59}$$

where we have introduced a notation $j_0 = J_0(\lambda_n)$, etc. Though $\lambda_n = 0$ is a solution of equation (59), we have proved earlier that it gives trivial eigensolutions and hence need not be considered.

For the product of two Bessel functions of integral order, we can write the following formula

$$J_m(\lambda) J_n(\lambda) = \left(\frac{\lambda}{2}\right)^{m+n} \sum_{r=0}^{\infty} \frac{(-1)^r (m+n+2r)!}{r!(m+r)!(n+r)!(m+n+r)!} \left(\frac{\lambda}{2}\right)^{2r}.$$

Substituting this in equation (59) and dropping a common factor λ_n, we get

$$\sum_{r=0}^{\infty} \frac{(-1)^r (2r)!}{(r!)^3 (r+1)!} \left[1 - 2(1-v)\frac{2r+1}{r+1}\right] \left(\frac{\lambda_n}{2}\right)^{2r} = 0 \tag{60}$$

as the equation for determining λ_n. The left-hand side is a polynomial of infinite degree in λ_n and has infinitely many roots λ_n. For v between 0 and 0·5 the term inside the bracket can be seen to be always negative. By substituting $\lambda_n = i\Lambda$ where Λ is real, we can show that the equation does not have any purely imaginary roots λ_n. Since the equation contains only even powers of λ_n with real coefficients, the roots will be complex conjugates occurring in groups of four, viz.:

$$\lambda_1 = a_1 + ib_1, \qquad \lambda_3 = -a_1 - ib_1,$$
$$\lambda_2 = a_1 - ib_1, \qquad \lambda_4 = -a_1 + ib_1.$$

For our semi-infinite cylinder we need only the roots with positive real part. For every such root λ_n we can get from equation (58b)

$$\frac{d_{1,n}}{d_{2,n}} = -\lambda_n \frac{j_0}{j_1}.$$

Hence from equations (57) the eigensolutions $f_{2n}(r)$ and $h_{2n}(r)$ can be written as

$$\begin{aligned} f_{2n}(r) &= -\lambda_n \frac{j_0}{j_1} J_0\left(\frac{\lambda_n r}{a}\right) + 4(1-\nu) J_0\left(\frac{\lambda_n r}{a}\right) - \lambda_n \frac{r}{a} J_1\left(\frac{\lambda_n r}{a}\right) \\ h_{2n}(r) &= -\lambda_n \frac{j_0}{j_1} J_1\left(\frac{\lambda_n r}{a}\right) + \frac{\lambda_n r}{a} J_0\left(\frac{\lambda_n r}{a}\right). \end{aligned} \tag{61a, b}$$

Summing up over all values of λ_n, the solutions (57) can be written in the form of infinite series containing a set of unknown scalar constants d_n,

$$\begin{aligned} u(x, r) &= \sum_{n=1}^{\infty} d_n f_{2n}(r)\, e^{-\lambda_n x/a} \\ w(x, r) &= \sum_{n=1}^{\infty} d_n h_{2n}(r)\, e^{-\lambda_n x/a}. \end{aligned} \tag{62a, b}$$

From boundary conditions (56) there follows that

$$r\left(1 - \frac{r}{a}\right) = \sum_{n=1}^{\infty} d_n f_{2n}(r) \qquad 0 = \sum_{n=1}^{\infty} d_n h_{2n}(r). \tag{63a, b}$$

These equations are equivalent to equations (36) which were written in matrix notation. From equations (53) an expression for the constants d_n can be written down at once:

$$d_n = \frac{1}{G_n} \int_0^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_0 \, dr \tag{64}$$

where $\mathbf{y}_n$ is obtained by substituting equations (61) into equations (46) and $\overline{\mathbf{S}}$ is given by equation (52b). We shall choose conveniently $\boldsymbol{\eta}_0(r)$ as zero and hence

$$\mathbf{y}_0 = \begin{bmatrix} \boldsymbol{\xi}_0(r) \\ \boldsymbol{\eta}_0(r) \end{bmatrix} = \begin{bmatrix} r(1-r/a) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

Then

$$\int_0^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_0 \, dr = \int_0^a \frac{\mu_1 r}{a^2} r(1-r/a) f_{2n}(r) \, dr.$$

Introducing $f_{2n}(r)$ from equation (61a), integrating and simplifying by using equation (59), we get

$$\int_0^a \mathbf{y}_n^T \overline{\mathbf{S}} \mathbf{y}_0 \, dr = \frac{\mu_1 a}{\lambda_n^2}\left[j_0 - 16\nu \frac{j_1}{\lambda_n} + \left\{ \frac{3}{\lambda_n} - \frac{j_1}{j_0} \right\} \left\{ \int_0^{\lambda_n} J_0(\rho) \, d\rho \right\} \right]. \tag{66}$$

Next, we find the normalizing constants by using equation (55). Guided by equation (61) for the eigensolutions, let us take two solutions $f_2(r, \lambda)$ and $h_2(r, \lambda)$ as a linear combination of the linearly independent solutions in the following form:

$$f_2(r, \lambda) = -\lambda\frac{J_0(\lambda)}{J_1(\lambda)}J_0\left(\frac{\lambda r}{a}\right)+4(1-\nu)J_0\left(\frac{\lambda r}{a}\right)-\frac{\lambda r}{a}J_1\left(\frac{\lambda r}{a}\right) \tag{67a}$$

$$h_2(r, \lambda) = -\lambda\frac{J_0(\lambda)}{J_1(\lambda)}J_1\left(\frac{\lambda r}{a}\right)+\frac{\lambda r}{a}J_0\left(\frac{\lambda r}{a}\right). \tag{67b}$$

These two functions obviously satisfy the differential equations (32) or (37a) and the conditions of finiteness at $r = 0$. Furthermore, when we substitute $\lambda = \lambda_n$ in these equations, they become identical to the eigensolutions f_{2n} and h_{2n} in equations (61). Introducing equations (61) and (67) into (55) and simplifying by using equation (59), we get the following simple expression for the normalizing constants:

$$G_n = 4(1-\nu)[4(1-\nu)j_1^2-2(1-2\nu)j_0^2]. \tag{68}$$

The constants d_n in equation (64) are completely determined when we evaluate equations (66) and (68). Then the complete solution for the displacements of the cylinder is given by equations (62). If the prescribed displacements are different from those considered in equation (56), only the integral in equation (64) needs to be reevaluated while the expression for the constants G_n remains the same as in equation (68).

We prescribed real displacements (56) and hence we expect the solutions (62) to be real. Though each term in the series on the right-handed sides in equations (62) is complex due to complex values of λ_n, their sum is expected to come out to be real. This can be easily proved as follows.

Since the eigenvalues λ_n will always occur in complex conjugate pairs, consider two eigenvalues λ_n and λ_m such that $\lambda_m = \bar{\lambda}_n$, where $\bar{\lambda}_n$ is the complex conjugate of λ_n. It is easy to see from equation (61) that $\boldsymbol{\xi}_m = \overline{\boldsymbol{\xi}}_n$ and then $\mathbf{y}_m = \overline{\mathbf{y}}_n$, $G_m = \overline{G}_n$, etc.

When the prescribed displacements are real, $\boldsymbol{\xi}_0(r)$ is real and since $\boldsymbol{\eta}_0(r)$ was chosen zero, also $\mathbf{y}_0(r)$ in equation (65) is real. Hence, from equation (64), the constants d_n and d_m will be complex conjugates also. It follows then that in the solutions (62) the series on the right-hand side are composed of an infinite number of terms occurring in complex conjugate pairs. Hence addition of all such terms will give only real displacements as required.

It is worthwhile to note the relative simplicity with which we were able to get an exact solution of this complex problem. The method presented can be used without difficulty to solve any general case. Problems concerning hollow cylinders under axisymmetric deformations can be solved completely by using the analytical results obtained here. For problems involving hollow or solid cylinders under general (non-axisymmetric) loading, the procedure is exactly the same and was already discussed before. In this case, to determine the auxiliary vector $\boldsymbol{\eta}(r)$ we make substitutions as in equations (39); the only difference being that the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ now each contain three elements instead of two and the matrices **P**, **Q**, **R**, **S** are 3×3 instead of 2×2. From thereon the procedure follows the same steps as illustrated in the axisymmetric problem and presents no other difficulties but only an increased amount of algebraic computations.

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REFERENCES

[1] L. POCHHAMMER, Beitrag zur Theorie der Biegung des Kreiszylinders, *Z. reine angew. Math.* **81**, 33–61 (1876).
[2] C. CHREE, The equations of an isotropic elastic solid in polar and cylindrical coordinates, their solution and application. *Camb. phil. Soc. Trans.* **14**, 250–369 (1889).
[3] L. N. G. FILON, On the elastic equilibrium of circular cylinders under certain practical systems of loads. *Phil. Trans. R. Soc.* **A198**, 147–233 (1902).
[4] A. I. LUR'E, *Three-Dimensional Problems of the Theory of Elasticity*, Moscow, 1955; English transl. by D. B. MCVEAN, pp. 380–439. Interscience (1964).
[5] G. HORVAY and J. A. MIRABAL, The end problem of cylinders. *J. appl. Mech.* **25**, 561–570 (1958).
[6] G. PICKETT, Application of the Fourier method to the solution of certain boundary problems in the theory of elasticity. *J. appl. Mech.* **11**, 176–182 (1944).
[7] W. E. WARREN and A. L. ROARK, A note on the end effect in isotropic cylinders. *AIAA Jnl* **5**, 1484–1486 (1967).
[8] G. A. BLISS, A boundary value problem for a system of ordinary linear differential equations of the first order. *Trans. Am. math. Soc.* **28**, 561–584 (1926).
[9] G. A. BLISS, Definitely self-adjoint boundary value problems. *Trans. Am. math. Soc.* **44–45**, 413–428 (1938–1939).
[10] K. MARGUERRE, *Handbook of Engineering Mechanics*, edited by W. FLÜGGE, pp. 33.1–33.26. McGraw-Hill (1962).
[11] G. E. LATTA, oral communications.

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Абстракт—Предметом этого анализа является однородный, изотропный, упругий, круглый цилиндр, подверженный действию некоторых сил или перемещений на своих поверхностях. Приводится общий метод решения, с помощью которого можно определить точно произвольные граничные условия заданные на кривых и плоских поверхностях полого или полного цилиндра произвольной длины. Граничные условия в перемещениях приводят к наиболее простому виду решения и, поэтому использованы для выяснения метода. Достаточно решить два фундаментальные задачи и далее использовать линейную суперпозицию для расчета каких либо других специфичных задач. В первой задаче рассматривается решение произвольных перемещений на кривых поверхностях бесконечного длинного цилиндра, тогда как во второй задаче решаются произвольные перемещения на плоском конце полубесконечного цилиндра, при нулевых перемещениях, появляющихся на его кривых поверхностях.